2

Homogeneous Coordinates and Transformations of the Plane

2.1 Introduction

In Chapter 1 planar objects were manipulated by applying one or more transformations. Section 1.7 identified the problem that the concatenation of a translation with a rotation, scaling or shear requires an awkward combination of a matrix addition and a matrix multiplication. The problem can be avoided by using an alternative coordinate system for which computations are performed by 3×3 matrix multiplications. Since

$$\begin{aligned} \begin{pmatrix} x' & y' & 1 \end{pmatrix} &= \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} a & d & 0 \\ b & e & 0 \\ c & f & 1 \end{pmatrix} \\ &= \begin{pmatrix} ax + by + c & dx + ey + f & 1 \end{pmatrix} \end{aligned} \tag{2.1}$$

it follows that

$$x' = ax + by + c \text{ and } y' = dx + ey + f \ .$$

To this end a new coordinate system is defined in which the point with Cartesian coordinates (x, y) is represented by the *homogeneous* or *projective* coordinates $(x, y, 1)$, or any multiple (rx, ry, r) with $r \neq 0$. The set of all homogeneous coordinates (x, y, w) is called the *projective plane* and denoted $\mathbb{P}^2$. In order to carry out transformations using matrix computations the homogeneous coordinates (x, y, w) are represented by the row matrix $(x \ y \ w)$. Equation (2.1) implies that any planar transformation can be performed by a 3×3 matrix multiplication and using homogeneous coordinates. Sometimes homogeneous coordinates will be denoted by capitals (X, Y, W) in order to distinguish them from the affine coordinates (x, y).

Example 2.1

1. $(1, 2, 3)$, $(2, 4, 6)$, and $(-1, -2, -3)$ are all homogeneous coordinates of the point $(1/3, 2/3)$ since
$$(1/3, 2/3, 1) = \frac{1}{3}(1, 2, 3) = \frac{1}{6}(2, 4, 6) = (-1)(-1, -2, -3) .$$
2. The Cartesian coordinates of the point with homogeneous coordinates $(X, Y, W) = (6, 4, 2)$ are obtained by dividing the coordinates through by $W = 2$ to give alternative homogeneous coordinates $(3, 2, 1)$. Thus the Cartesian coordinates of the point are $(x, y) = (3, 2)$.

EXERCISES

2.1. Which of the following homogeneous coordinates $(2, 6, 2)$, $(2, 6, 4)$, $(1, 3, 1)$, $(-1, -3, -2)$, $(1, 3, 2)$, and $(4, 12, 8)$ represent the point $(1/2, 3/2)$?

2.2. Write down two sets of homogeneous coordinates of $(2, -3)$.

2.3. A point has Cartesian coordinates $(5, -20)$ and homogeneous coordinates $(-5, ?, -1)$ and $(10, -40, ?)$. Fill in the missing entries indicated by a "?".

Definition 2.2

A *(projective) transformation* of the projective plane is a mapping $L : \mathbb{P}^2 \to \mathbb{P}^2$ of the form

$$L(x, y, w) = \begin{pmatrix} x & y & w \end{pmatrix} \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & k \end{pmatrix} \quad (2.2)$$
$$= (ax + by + cw, dx + ey + fw, gx + hy + kw) , \quad (2.3)$$

for some constant real numbers a, b, c, d, e, f, g, h, k. A matrix which represents a linear transformation of the projective plane is called a *homogeneous transformation matrix*. When $g = h = 0$ and $k \neq 0$, L is said to be an *affine* *transformation*. Affine transformations correspond to transformations of the Cartesian plane.

Remark 2.3

If alternative homogeneous coordinates (rx, ry, rw) are taken in (2.2) then

$$L(rx, ry, rw) = (arx + bry + crw, drx + ery + frw, grx + hry + krw) ,$$

and dividing through by r gives the homogeneous coordinates (2.3). Thus $L(rx, ry, rw)$ and $L(x, y, w)$ map to the same point, and therefore the definition of a transformation does not depend on the choice of homogeneous coordinates for a given point.

2.1.1 Homogeneous Coordinates

A more formal definition of homogeneous coordinates is obtained in terms of an equivalence relation.

Definition 2.4

A *relation* $\sim$ on a set S is a rule which determines whether two members of the set S are considered related or not. If s_1 is related to s_2, then this is expressed by writing $s_1 \sim s_2$.

Example 2.5

"Greater than", with its usual meaning, is a relation on $\mathbb{R}$. The relationship "3 is greater than 2" is written $3 \sim 2$. The relation "greater than" is generally written $3 > 2$ where the symbol $\sim$ is substituted by $>$. The number 2 is not related to 3 since *it is not true* that $2 > 3$.

Definition 2.6

A relation $\sim$ on a set S is said to be

1. *reflexive* if $s \sim s$ for all s in S;
2. *symmetric* if whenever $s_1 \sim s_2$, then $s_2 \sim s_1$;
3. *transitive* if whenever $s_1 \sim s_2$ and $s_2 \sim s_3$, then $s_1 \sim s_3$;
4. an *equivalence relation* if $\sim$ is reflexive, symmetric, and transitive.

Example 2.7

The relation $>$ on $\mathbb{R}$ is transitive, but not reflexive or symmetric. The relations $\geq$ and $\leq$ are both reflexive and transitive, but not symmetric. The most familiar equivalence relation on $\mathbb{R}$ is $=$.

Definition 2.8

Let s_1 be a member of S. The subset of S, consisting of every s in S which is related to s_1, is called the *equivalence class* of s_1 and denoted by $[s_1]$. A member of an equivalence class $[s_1]$ is called a *representative* of $[s_1]$. Clearly, if s is a representative of $[s_1]$ then $s \sim s_1$.

Homogeneous coordinates arise as equivalence classes determined by the following lemma which defines an equivalence relation on $S = \mathbb{R}^3 \backslash \{(0,0,0)\}$ (that is, S consists of all $\mathbb{R}^3$ excluding the origin).

Lemma 2.9

The relation $\sim$ on the set $S = \mathbb{R}^3 \backslash \{(0,0,0)\}$ defined by

$$(x_0, y_0, w_0) \sim (x_1, y_1, w_1) \Leftrightarrow (x_1, y_1, w_1) = r(x_0, y_0, w_0) \text{ for some } r \neq 0$$

is an equivalence relation.

Proof

1. The relation $\sim$ is reflexive since $(x_0, y_0, w_0) = 1(x_0, y_0, w_0)$.
2. The relation $\sim$ is symmetric since if $(x_0, y_0, w_0) \sim (x_1, y_1, w_1)$, then $(x_1, y_1, w_1) = r(x_0, y_0, w_0)$ for some $r \neq 0$. Thus $(x_0, y_0, w_0) = \frac{1}{r}(x_1, y_1, w_1)$, and hence $(x_1, y_1, w_1) \sim (x_0, y_0, w_0)$.
3. Suppose $(x_0, y_0, w_0) \sim (x_1, y_1, w_1)$, and $(x_1, y_1, w_1) \sim (x_2, y_2, w_2)$. Then $(x_1, y_1, w_1) = r_1(x_0, y_0, w_0)$ for some $r_1 \neq 0$, and $(x_2, y_2, w_2) = r_2(x_1, y_1, w_1)$ for some $r_2 \neq 0$. So

   $$(x_2, y_2, w_2) = r_2(x_1, y_1, w_1) = r_2 r_1(x_0, y_0, w_0), \text{ for } r_2 r_1 \neq 0 ,$$

   and hence $(x_2, y_2, w_2) \sim (x_0, y_0, w_0)$. Hence $\sim$ is transitive.

□

The equivalence classes $[(x, y, w)]$ are the sets

$$[(x, y, w)] = \{ \ r(x, y, w) \mid r \in \mathbb{R}, r \neq 0 \ \} .$$

The projective plane $\mathbb{P}^2$ is defined to be the set of all equivalence classes. An equivalence class is referred to as a *point* of the projective plane.

In practice, operations of the projective plane are carried out by taking a representative for each equivalence class. Homogeneous coordinates (X, Y, W) with $W \neq 0$ have a representative of the form $(x, y, 1)$ where $x = X/W$, and $y = Y/W$. Thus there is a $1-1$ correspondence between points (x, y) of the Cartesian plane and points (X, Y, W) in the projective plane with $W \neq 0$. Points with $W = 0$ are discussed in Section 2.2. Then, a transformation is a mapping of equivalence classes, that is, a mapping of points in the projective plane. Remark 2.3 states that the definition of a transformation does not depend on the choice of the representative of an equivalence class.

Exercise 2.4

Define a relation $\sim$ on non-singular 3×3 matrices by $M_1 \sim M_2$ if and only if $M_1 = \mu M_2$ for some $\mu \neq 0$. Show that $\sim$ is an equivalence relation.

2.2 Points at Infinity

Homogeneous coordinates of the form $(x, y, 0)$ do not correspond to a point in the Cartesian plane, but represent the unique *point at infinity* in the direction $(x\ y)$. To justify this remark, consider the line $(x(t), y(t)) = (tx + a, ty + b)$ through the point (a, b) with direction $(x\ y)$. The point $(tx + a, ty + b)$ has homogeneous coordinates $(tx + a, ty + b, 1)$ and multiplying through by $1/t$ (for $t \neq 0$) gives alternative homogeneous coordinates $(x + a/t, y + b/t, 1/t)$. Points on the line an infinite distance away from the origin in the Cartesian plane may be obtained by letting t tend to infinity. The limiting point of $(x + a/t, y + b/t, 1/t)$ as $t \to \infty$ is $(x, y, 0)$. Therefore, it is natural to interpret the homogeneous coordinates $(x, y, 0)$ as the point at infinity in the direction (x, y). The projective plane may be interpreted as the Cartesian plane together with all the points at infinity.

The projective plane also makes sense of the intuitive notion that two parallel lines intersect at infinity. For instance, consider the parallel lines

$$x + 2y = 1\ , \text{ and} \tag{2.4}$$

$$x + 2y = 2\ . \tag{2.5}$$

Let (X, Y, W) be homogeneous coordinates of a point (x, y) on the line (2.4). Then $(x, y) = (X/W, Y/W)$ and hence

$$(X/W) + 2\,(Y/W) = 1\ .$$

Multiplying through by W, yields the *homogeneous equation* of the line

$$X + 2Y = W \ . \tag{2.6}$$

Similarly, the homogeneous equation of (2.5) is

$$X + 2Y = 2W \ . \tag{2.7}$$

Equations (2.6) and (2.7) have common solutions of the form $(-2r, r, 0)$. The solutions are all homogeneous coordinates of the point $(-2, 1, 0)$ which is the unique point of intersection of the parallel lines. It is easily verified that $(-2, 1, 0)$ is the point at infinity in the direction of the lines. A similar argument yields that all parallel lines intersect in a unique point at infinity.

EXERCISES

2.5. Find the point at infinity in the direction of the vector $(6, -3)$.

2.6. Find the point at infinity on the line $4x - 3y + 1 = 0$.

2.7. Determine the homogeneous equation of the line $3x + 4y = 5$.

2.8. Determine the homogeneous coordinates of the point at infinity which is the intersection of the lines $2x - 9y = 5$ and $2x - 9y = 7$. Verify that the intersection is the point at infinity in the direction of the lines.

2.9. Determine the point at infinity on the line $ax + by + c = 0$. Conclude that all lines in the direction $(-b, a)$ intersect in a unique point at infinity.

2.3 Visualization of the Projective Plane

There are two models that interpret homogeneous coordinates geometrically, and hence enable the projective plane to be visualized.

2.3.1 Line Model of the Projective Plane

The line model of the projective plane is obtained by representing the point with homogeneous coordinates $\mu(X, Y, W)$, $\mu \neq 0$, by the line through the origin with direction (X, Y, W) in (X, Y, W)-space. Since the point with Cartesian coordinates (x, y) has homogeneous coordinates of the form $(X, Y, W) = r(x, y, 1)$

for $r \neq 0$, there is a $1-1$ correspondence between points (x, y) of the Cartesian plane and the lines

$$\{ \ r(x, y, 1) \mid r \in \mathbb{R} \ \} \tag{2.8}$$

as illustrated in Figure 2.1. There is also a $1-1$ correspondence between the points (x, y) and the points $(x, y, 1)$ of the $W = 1$ plane.

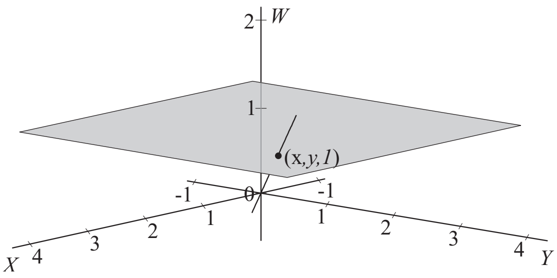

Figure 2.1 The line model of the projective plane

The $W = 1$ plane is inadequate for studying the projective plane since points at infinity do not correspond to points in the $W = 1$ plane, nor to lines of the form (2.8). Instead, points at infinity correspond to lines in the $W = 0$ plane. For example, the parallel lines (2.4) and (2.5) correspond to the planes in (X, Y, W)-space defined by Equations (2.6) and (2.7). The planes intersect in a line through the origin in the $W = 0$ plane as shown in Figure 2.2. The line is parametrized by $(-2t, t, 0)$ and corresponds to the point at infinity $(-2, 1, 0)$ which is the intersection of the two parallel lines. The difficulty with the line model is that lines in the projective plane correspond to planes in the model, and more generally, curves in the projective plane correspond to surfaces. To visualize curves in the projective plane the spherical model is introduced.

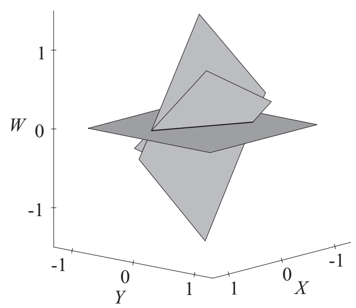

Figure 2.2 Intersection of planes corresponding to parallel lines in the Cartesian plane

2.3.2 Spherical Model of the Projective Plane

The spherical model of the projective plane is obtained by representing the point with homogeneous coordinates $\mu(X, Y, W)$, $\mu \neq 0$, by the points of intersection of the line through the origin with direction (X, Y, W) and the unit sphere centred at the origin $X^2 + Y^2 + W^2 = 1$ as illustrated in Figure 2.3. The intersections are the antipodal points

$$\pm\left(\frac{X}{X^2 + Y^2 + W^2}, \frac{Y}{X^2 + Y^2 + W^2}, \frac{W}{X^2 + Y^2 + W^2}\right) .$$

Since antipodal points on the sphere correspond to the same point in the projective plane, it suffices to consider the upper half-sphere together with (half of) the equator. (The equator is the circle of intersection of the sphere with the $W = 0$ plane.) Points at infinity $(X, Y, 0)$ correspond to points on the equator.

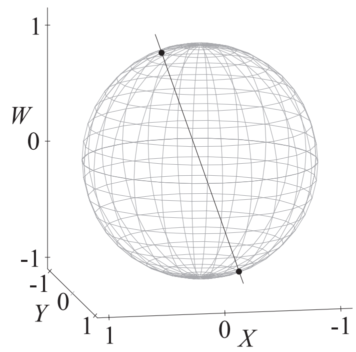

Figure 2.3 Spherical model of the projective plane. Antipodal points represent the same homogeneous point.

Thus the sphere provides a way of visualizing *all* homogeneous coordinates. For instance, the intersection of parallel lines can be visualized in the spherical model. Lines in the Cartesian plane correspond to planes which intersect the sphere in a great circle. The intersection of two parallel lines corresponds to the intersection of the two great circles on the sphere, namely, two antipodal points at infinity on the equator. Figure 2.4 shows how two great circles, representing the lines (2.4) and (2.5), intersect in the antipodal points $(-2/\sqrt{5}, 1/\sqrt{5}, 0)$ and $(2/\sqrt{5}, -1/\sqrt{5}, 0)$ on the equator.

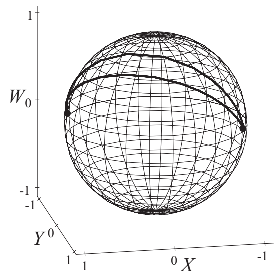

Figure 2.4 Intersection of parallel lines on the spherical model of the projective plane

2.4 Transformations in Homogeneous Coordinates

In the following sections the homogeneous transformation matrices for translations, scalings, and rotations are described. In order to minimize on notation, a transformation and its homogeneous transformation matrix will be given the same notation. For instance, a translation and its translation matrix are both denoted $\mathsf{T}(h,k)$.

2.4.1 Translations

The homogeneous translation matrix for the translation $\mathsf{T}(h,k)$ is

$$\mathsf{T}(h,k) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h & k & 1 \end{pmatrix}.$$

Then

$$\begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h & k & 1 \end{pmatrix} = \begin{pmatrix} x+h & y+k & 1 \end{pmatrix},$$

verifying that the point (x,y) is translated to $(x+h, y+k)$.

Example 2.10

In Example 1.8 the translation $\mathsf{T}(2,1)$ was applied to the quadrilateral with vertices $\mathbf{A}(1,1)$, $\mathbf{B}(3,1)$, $\mathbf{C}(2,2)$, and $\mathbf{D}(1.5,3)$. Let the homogeneous coordinates of the 4 vertices be expressed as the rows of a 4×3 matrix. The translation is applied by multiplying the matrix of vertices by the translation matrix. The rows of the resulting matrix are the homogeneous coordinates of images of the vertices.

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \\ 2 & 2 & 1 \\ 1.5 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 5 & 2 & 1 \\ 4 & 3 & 1 \\ 3.5 & 4 & 1 \end{pmatrix} .$$

$$\text{vertices} \quad \times \quad \text{translation} \quad = \quad \text{images of vertices}$$

The images have Cartesian coordinates $\mathbf{A}'(3,2)$, $\mathbf{B}'(5,2)$, $\mathbf{C}'(4,3)$, and $\mathbf{D}'(3.5,4)$.

2.4.2 Scaling about the Origin

The homogeneous scaling matrix is

$$\mathsf{S}(s_x, s_y) = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Then

$$\begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} s_x x & s_y y & 1 \end{pmatrix} ,$$

verifying that the point $(x, y, 1)$ is mapped to $(s_x x, s_y y, 1)$. The scaling can also be performed by the scaling matrix

$$\mathsf{S}(s_x, s_y; s_w) = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_w \end{pmatrix}$$

for $s_w \neq 0$. The transformation $\mathsf{S}(s_x, s_y; s_w)$ represents a scaling about the origin by a factor of s_x/s_w in the x-direction, and by a factor of s_y/s_w in the y-direction. The semicolon before the s_w is used to distinguish the planar scaling from the spatial scaling which is introduced in Chapter 3.

Example 2.11

A scaling about the origin by a factor of 4 in the x-direction, and by a factor of 2 in the y-direction, of the unit square with vertices $(1,1)$, $(2,1)$, $(2,2)$, and $(1,2)$ is determined by

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 1 \\ 8 & 2 & 1 \\ 8 & 4 & 1 \\ 4 & 4 & 1 \end{pmatrix} .$$

The image is a square with vertices $(4,2)$, $(8,2)$, $(8,4)$, and $(4,4)$.

2.4.3 Rotation about the Origin

In homogeneous coordinates the transformation matrix for a rotation $\mathsf{Rot}\,(\theta)$ about the origin through an angle θ is

$$\mathsf{Rot}\,(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where a positive angle denotes an anticlockwise rotation. Hence

$$\begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta & x\sin\theta + y\cos\theta & 1 \end{pmatrix}.$$

Example 2.12

An anticlockwise rotation about the origin through an angle $\pi/3$ of the unit square with vertices $(1,1)$, $(2,1)$, $(2,2)$, and $(1,2)$ is determined by

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0.5 & 0.866 & 0 \\ -0.866 & 0.5 & 0 \\ 0 & 0 & 1.0 \end{pmatrix} = \begin{pmatrix} -0.366 & 1.366 & 1.0 \\ 0.134 & 2.232 & 1.0 \\ -0.732 & 2.732 & 1.0 \\ -1.232 & 1.866 & 1.0 \end{pmatrix}.$$

The image is a square with vertices $(-0.366, 1.366)$, $(0.134, 2.232)$, $(-0.732, 2.732)$, and $(-1.232, 1.866)$.

EXERCISES

2.10. Apply the translation $\mathsf{T}\,(-2,-1)$ to the quadrilateral, obtained in Example 2.10, with vertices $\mathbf{A}'(3,2)$, $\mathbf{B}'(5,2)$, $\mathbf{C}'(4,3)$, and $\mathbf{D}'(3.5,4)$.

2.11. Write down the transformation matrix which has the effect of a scaling by a factor of 2 in the x-direction and by a factor of 1.5 in the y-direction. Apply the transformation to the quadrilateral of Example 2.10. Compare the result with Example 1.11.

2.12. Write down the transformation matrix which has the effect of an anticlockwise rotation about the origin through an angle $\pi/2$. Apply the transformation to the quadrilateral of Example 2.10.

2.13. Determine the matrix for the inverse scaling transformation of Exercise 2.11.

2.14. Determine the homogeneous transformation matrix of $\mathsf{Rot}(\theta)^{-1}$.

2.15. Determine the homogeneous transformation matrices for reflections in the x- and y-axes.

2.5 Concatenation of Transformations

In homogeneous coordinates, the concatenation of transformations T_1 and T_2, denoted $T_1 \circ T_2$, can be performed with matrix multiplications alone. For example, a rotation $\mathsf{Rot}(\theta)$ about the origin followed by a translation $\mathsf{T}(h,k)$ is denoted $\mathsf{Rot}(\theta) \circ \mathsf{T}(h,k)$, and has the homogeneous transformation matrix

$$\begin{aligned}\mathsf{Rot}(\theta)\,\mathsf{T}(h,k) &= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h & k & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ h & k & 1 \end{pmatrix}.\end{aligned}$$

Example 2.13

The transformation matrix which represents an anticlockwise rotation of $3\pi/2$ about the origin followed by a scaling by a factor of 3 units in the x-direction and 2 units in the y-direction is

$$\mathsf{Rot}(3\pi/2)\,\mathsf{S}(3,2) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

EXERCISES

2.16. Determine the matrix which represents the operations of Example 2.13 performed in *reverse order*. What can be deduced about the order in which transformations are performed?

2.17. Determine the matrix which represents an anticlockwise rotation about the origin through an angle π followed by a scaling by a factor of 4 in the x-direction and by a factor of 0.5 in the y-direction.

2.18. Determine the matrix which represents a translation of 4 units in the x-direction followed by a rotation about the point $(2, 3)$ through an angle $\pi/2$ in a *clockwise* direction.

2.5.1 Inverse Transformations

The *identity* and *inverse* transformations were introduced in Section 1.2. The *identity transformation* I is the transformation which has the effect of leaving all points of the plane unchanged. The *inverse* of a transformation L, denoted L^{-1}, has the effect of mapping images of the transformation L back to their original points. These transformations can be given a more precise definition in terms of the concatenation of transformations.

Definition 2.14

The *identity* transformation of the plane, denoted I, is the transformation for which $I \circ L = L \circ I = L$, for all planar transformations L. The transformation matrix of the identity transformation is the 3×3 identity matrix I_3 (that is, the matrix with values of 1's on the leading diagonal and 0's elsewhere).

Definition 2.15

The inverse L^{-1} of a transformation L is the transformation such that $L \circ L^{-1} = I$ and $L^{-1} \circ L = I$.

Lemma 2.16

Let the homogeneous transformation matrix of L be T. A necessary and sufficient condition for the inverse L^{-1} to exist is that T^{-1} exists and is the transformation matrix of L^{-1}.

Proof

Suppose L has an inverse L^{-1} with transformation matrix T_1. The concatenation $L \circ L^{-1} = I$ has transformation matrix $\mathrm{T}\mathrm{T}_1 = I_3$. Similarly, $L^{-1} \circ L = I$ has transformation matrix $\mathrm{T}_1\mathrm{T} = I_3$. Thus by the definition of a matrix inverse $\mathrm{T}_1 = \mathrm{T}^{-1}$.

Conversely, suppose T has an inverse T^{-1}, and let L_1 be the transformation defined by T^{-1}. Since $\mathrm{T}\mathrm{T}^{-1} = I_3$ and $\mathrm{T}^{-1}\mathrm{T} = I_3$ it follows that $L \circ L_1 = I$ and $L_1 \circ L = I$. Hence L_1 is the inverse transformation of L. □

Definition 2.17

A transformation $L : \mathbb{P}^2 \to \mathbb{P}^2$ which has an inverse L^{-1} is called a *non-singular transformation*. Lemma 2.16 shows that a transformation is a non-singular transformation if and only if its transformation matrix is non-singular.

Example 2.18

Non-singular matrices A and B satisfy $(AB)^{-1} = B^{-1}A^{-1}$. Further, $\mathsf{S}\,(s_1, s_2)^{-1} = \mathsf{S}\,(1/s_1, 1/s_2)$ and $\mathsf{Rot}\,(\theta)^{-1} = \mathsf{Rot}\,(-\theta)$ (Exercises 1.9 and 1.14). This gives a straightforward way of determining the inverse transformation matrix of the concatenated transformation $\mathsf{Rot}\,(3\pi/2) \circ \mathsf{S}\,(3,2)$:

$$\begin{aligned}
(\mathsf{Rot}\,(3\pi/2)\,\mathsf{S}\,(3,2))^{-1} &= \mathsf{S}\,(3,2)^{-1}\,\mathsf{Rot}\,(3\pi/2)^{-1} \\
&= \mathsf{S}\,(1/3,1/2)\,\mathsf{Rot}\,(-3\pi/2) \\
&= \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\left(-\frac{3\pi}{2}\right) & \sin\left(-\frac{3\pi}{2}\right) & 0 \\ -\sin\left(-\frac{3\pi}{2}\right) & \cos\left(-\frac{3\pi}{2}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \frac{1}{3} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} .
\end{aligned}$$

Alternatively, using Example 2.13

$$(\mathsf{Rot}\,(3\pi/2)\,\mathsf{S}\,(3,2))^{-1} = \begin{pmatrix} 0 & -2 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \frac{1}{3} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

EXERCISES

2.19. Determine the transformation matrix of the inverse of the concatenation $\mathsf{T}\,(-2,5) \circ \mathsf{Rot}\,(-\pi/3)$.

2.20. Use a graphics calculator or mathematics computer package to compute the inverse of the transformation with matrix

$$\begin{pmatrix} 1.0 & 0.5 & 0.0 \\ 0.8 & -1.2 & 0.0 \\ 4.0 & -2.0 & 1.0 \end{pmatrix} .$$

2.21. Consider a (rectangular) Cartesian coordinate system with origin $\mathbf{O}$ and coordinates (x, y), and a second system with origin $\mathbf{O}'(x_0, y_0)$

and coordinates (x', y'). The origin and axes of the first system can be mapped to those of the second by applying a rotation $\mathsf{Rot}(\theta)$ followed by the translation $\mathsf{T}(x_0, y_0)$. The (x, y)-coordinates of a point given in (x', y')-coordinates is obtained by applying the *orthogonal change of coordinates* transformation

$$\begin{aligned} x &= x' \cos\theta - y' \sin\theta + x_0 \\ y &= x' \sin\theta + y' \cos\theta + y_0 \,. \end{aligned}$$

a) Determine the homogeneous transformation matrix A of the change of coordinates and show that $\det(\mathsf{A}) = 1$.

b) Determine the inverse change of coordinates transformation which determines the (x', y')-coordinates of a point (x, y).

c) Show that a change of coordinates preserves the angle between a pair of lines.

d) Show that the x'- and y'-axes, expressed in (x, y)-coordinates, are given by the equations

$$\begin{aligned} (x - x_0)\sin\theta - (y - y_0)\cos\theta &= 0 \,, \quad \text{and} \\ (x - x_0)\cos\theta + (y - y_0)\sin\theta &= 0 \,. \end{aligned}$$

2.5.2 Rotation about an Arbitrary Point

A rotation through an angle θ about an arbitrary point (x_0, y_0) is obtained by performing a translation which maps (x_0, y_0) to the origin, followed by a rotation through an angle θ about the origin, and followed by a translation which maps the origin to (x_0, y_0). The rotation matrix is

$$\begin{aligned} \mathsf{Rot}_{(x_0,y_0)}(\theta) &= \mathsf{T}(-x_0, -y_0)\,\mathsf{Rot}(\theta)\,\mathsf{T}(x_0, y_0) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_0 & -y_0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_0 & y_0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ \begin{matrix}(-x_0\cos\theta \\ +y_0\sin\theta + x_0)\end{matrix} & \begin{matrix}(-x_0\sin\theta \\ -y_0\cos\theta + y_0)\end{matrix} & 1 \end{pmatrix} . \end{aligned}$$

Example 2.19

A square has vertices $\mathbf{A}(1,1)$, $\mathbf{B}(2,1)$, $\mathbf{C}(2,2)$, and $\mathbf{D}(1,2)$. Calculate the coordinates of the vertices when the rectangle is rotated about $\mathbf{B}$ through an angle $\pi/4$. The required transformation is

$$\begin{aligned}
&\mathsf{T}\,(-2,-1)\,\mathsf{Rot}\,(\pi/4)\,\mathsf{T}\,(2,1)\\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0.7071 & 0.7071 & 0 \\ -0.7071 & 0.7071 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}\\
&= \begin{pmatrix} 0.7071 & 0.7071 & 0 \\ -0.7071 & 0.7071 & 0 \\ 1.2929 & -1.1213 & 1 \end{pmatrix}.
\end{aligned}$$

Applying the transformation to the vertices,

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0.7071 & 0.7071 & 0 \\ -0.7071 & 0.7071 & 0 \\ 1.2929 & -1.1213 & 1 \end{pmatrix} = \begin{pmatrix} 1.2929 & 0.2929 & 1 \\ 2 & 1 & 1 \\ 1.2929 & 1.7071 & 1 \\ 0.5858 & 1 & 1 \end{pmatrix}$$

gives $\mathbf{A}'(1.2929, 0.2929)$, $\mathbf{B}'(2,1)$, $\mathbf{C}'(1.2929, 1.7071)$, and $\mathbf{D}'(0.5858, 1.0)$. The rotated square is illustrated in Figure 2.5.

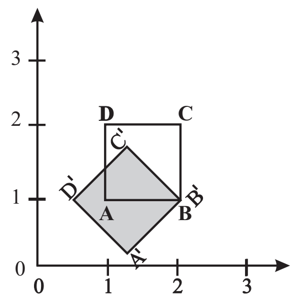

Figure 2.5

2.5.3 Reflection in an Arbitrary Line

Reflections in the x- and y-axes were derived in Exercise 2.15.A reflection in an arbitrary line ℓ with equation $ax + by + c = 0$ is obtained by transforming the line to one of the axes, reflecting in that axis, and then applying the inverse of the first transformation. Suppose $b \neq 0$.

1. The line ℓ intersects the y-axis in the point $(0, -c/b)$.
2. Apply a translation mapping $(0, -c/b)$ to the origin, and thus mapping ℓ to a line ℓ' through the origin with an identical gradient to ℓ.
3. The gradient of ℓ' is $\tan\theta = -a/b$, where θ is the angle that ℓ makes with the x-axis. Rotate ℓ' about the origin through an angle $-\theta$. The line is now mapped to the x-axis.
4. Apply a reflection in the x-axis.
5. Apply the inverse of the rotation of step 3, followed by the inverse of the translation of step 2.

The concatenation of the above transformations is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c/b & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c/b & 1 \end{pmatrix} \tag{2.9}$$

$$= \begin{pmatrix} \cos^2\theta - \sin^2\theta & 2\cos\theta\sin\theta & 0 \\ 2\cos\theta\sin\theta & \sin^2\theta - \cos^2\theta & 0 \\ 2\frac{c}{b}\sin\theta\cos\theta & \frac{c}{b}\left(\sin^2\theta - \cos^2\theta - 1\right) & 1 \end{pmatrix}. \tag{2.10}$$

Since $\tan\theta = \sin\theta/\cos\theta = -a/b$, it follows that $\sin\theta = a/\left(a^2+b^2\right)^{1/2}$ and $\cos\theta = -b/\left(a^2+b^2\right)^{1/2}$ (Exercise 2.25). Hence, $\cos^2\theta = b^2\big/\left(a^2+b^2\right)$, $\sin^2\theta = a^2\big/\left(a^2+b^2\right)$, $\sin\theta\cos\theta = -ab\big/\left(a^2+b^2\right)$, and $\cos^2\theta - \sin^2\theta = \left(b^2-a^2\right)\big/\left(a^2+b^2\right)$. Finally, substitution for the trigonometric functions in (2.10) yields

$$\begin{pmatrix} \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} & 0 \\ -\frac{2ab}{a^2+b^2} & -\frac{b^2-a^2}{a^2+b^2} & 0 \\ -\frac{2ac}{a^2+b^2} & -\frac{2bc}{a^2+b^2} & 1 \end{pmatrix}.$$

Since in homogeneous coordinates multiplication by a factor does not affect the result, the above matrix can be multiplied by a factor $\left(a^2+b^2\right)$ to give the general reflection matrix

$$\mathsf{R}_{(a,b,c)} = \begin{pmatrix} b^2-a^2 & -2ab & 0 \\ -2ab & -b^2+a^2 & 0 \\ -2ac & -2bc & a^2+b^2 \end{pmatrix}. \tag{2.11}$$

EXERCISES

2.22. Show that the concatenation of two rotations, the first through an angle θ about a point $\mathbf{P}(x_0, y_0)$ and the second about a point $\mathbf{Q}(x_1, y_1)$ (distinct from $\mathbf{P}$) through an angle $-\theta$, is equivalent to a translation.

2.23. Determine the transformation matrix of a reflection in the line $5x - 2y + 8 = 0$. Express the reflection first using (2.11) and then as a concatenation of transformations (2.9).

2.24. Demonstrate that if the coordinates of points are expressed by rational numbers (whole numbers and fractions), then a reflection in a line defined by rational coefficients a, b, c can be computed using integer arithmetic.

2.25. Use trigonometry to verify the result used in the derivation of (2.11) that if $\tan\theta = -a/b$, then $\sin\theta = a/\left(a^2+b^2\right)^{1/2}$ and $\cos\theta = -b/\left(a^2+b^2\right)^{1/2}$.

2.6 Applications

2.6.1 Instancing

In Section 1.8.1 the model of the front of a house was defined by instancing the picture element **Square** with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$. The front door was obtained by applying a scaling of 0.5 units in the x-direction, followed by a translation of 3 units in the x-direction and 1 unit in the y-direction. Transformations applied to picture elements and primitives to obtain instances are called *modelling transformations.* The front door is obtained from **Square** by applying the modelling transformation $\mathsf{S}(1,3) \circ \mathsf{T}(4,0)$ which has the *modelling transformation matrix*

$$\mathsf{S}(1,3)\mathsf{T}(4,0) = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}.$$

The vertices of the door are obtained by applying the modelling transformation matrix to the vertices of the **Square** primitive, giving

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 1 \\ 4 & 1 & 1 \\ 4 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix}.$$

So in world coordinates the vertices are $(3,1)$, $(4,1)$, $(4,2)$ and $(3,2)$.

Exercise 2.26

Determine the modelling transformation matrices of the four instances of **Square** which define the windows of the front of the house in Figure 1.8. Complete the picture element **House** by determining the modelling transformation matrix of the primitive **Point** which is a small circle centred at the point $(0, 0)$. Now create a modern housing estate by instancing **House**!

2.6.2 Device Coordinate Transformation

Sections 1.8.1 and 2.6.1 discuss how the model of an object is obtained by instancing a number of picture elements and graphical primitives. The object (the front of a house) is defined in a two-dimensional world coordinate system. The object is displayed in a device window, such as a computer screen, by applying a *device coordinate transformation*. The process of viewing an object defined in a *three*-dimensional world coordinate system is discussed later in Chapter 4.

Suppose the world coordinate system is the (x, y)-plane. The region of the plane to be displayed by the device is specified by a rectangular *window* with lower left corner $(x_{\min}, y_{\min})$ and upper right corner $(x_{\max}, y_{\max})$. Any part of the object lying outside this region is "clipped" and is not displayed. The coordinate system of a display device is determined by its resolution. For example, a computer screen consists of a rectangular array of pixels. The number of pixels in the horizontal (h) and vertical (v) directions is written $h \times v$ and called the screen resolution. The origin is assumed to be the lower left corner of the screen, and the pixels are labelled with coordinates (h, v) where h and v are non-negative integers. Figure 2.6 illustrates a screen with a resolution of 1280×1024 pixels, and a window given by $(x_{\min}, y_{\min}) = (-10, -5)$ and $(x_{\max}, y_{\max}) = (10, 5)$. The window is mapped onto the screen by the device coordinate transformation which is the concatenation of (i) the translation $\mathsf{T}(10, 5)$ which maps the point $(-10, -5)$ to the origin, and (ii) a scaling $\mathsf{S}(1280/20, 1024/10)$ which makes the rectangle the same size as the screen. Therefore, the device coordinate transformation is

$$\begin{aligned} \mathsf{T}(10,5)\,\mathsf{S}(1280/20, 1024/10) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 10 & 5 & 1 \end{pmatrix} \begin{pmatrix} 64 & 0 & 0 \\ 0 & 102.4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 64 & 0 & 0 \\ 0 & 102.4 & 0 \\ 640 & 512 & 1 \end{pmatrix} . \end{aligned}$$

Hence the Cartesian coordinates of the point on the screen corresponding to the point (x, y) in the window are $(64x + 640, 102.4y + 512)$.

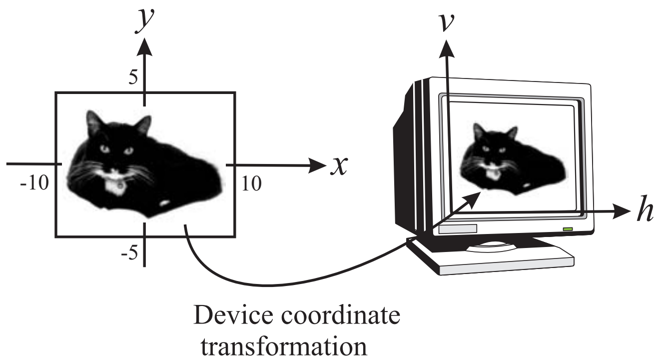

Figure 2.6 A device coordinate transformation makes Max a film star

Exercise 2.27

Suppose the window specified above is to be mapped onto a rectangular *device window* of the computer screen with lower left corner $(200, 200)$ and upper right corner $(600, 400)$. Determine the device coordinate transformation matrix.

2.7 Point and Line Geometry in Homogeneous Coordinates

The general equation of a line in the Cartesian plane is $ax + by + c = 0$. Suppose (X, Y, W) are the homogeneous coordinates of the point (x, y), so that $x = X/W$ and $y = Y/W$. Substituting for x and y in the equation of the line, and multiplying through by W, yields the condition for (X, Y, W) to be a point on the line

$$aX + bY + cW = 0 . \tag{2.12}$$

The equation is known as the *homogeneous line equation*. The line is uniquely defined by the coefficients a, b, and c, or any non-zero multiple ra, rb, and rc of them. Therefore, it is natural to specify the line by the homogeneous *line coordinates*

$$\ell = (a, b, c) \ .$$

It is also useful to consider ℓ to be a vector known as the *line vector*. Since any non-zero multiple of ℓ defines the same line, only the direction of ℓ is of importance. Let $\mathbf{P}(X, Y, W)$ be a point on the line. By permitting the homogeneous coordinates (X, Y, W) to be treated as a vector, Equation (2.12) may be expressed as the dot product

$$\ell \cdot \mathbf{P} = aX + bY + cW = 0\ . \tag{2.13}$$

The identity (2.13) leads to two useful operations: (i) determining the line through two distinct points, and (ii) determining the point of intersection of two lines.

To Find the Equation of the Line Through Two Points

Suppose ℓ is the line vector of a line containing two distinct points $\mathbf{P}_1(X_1, Y_1, W_1)$ and $\mathbf{P}_2(X_2, Y_2, W_2)$. Then (2.13) yields

$$\ell \cdot \mathbf{P}_1 = 0 \quad \text{and} \quad \ell \cdot \mathbf{P}_2 = 0\ .$$

For any two vectors, the condition $\mathbf{a} \cdot \mathbf{b} = 0$ implies that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular. Hence, ℓ is a vector perpendicular to both $\mathbf{P}_1$ and $\mathbf{P}_2$. To determine ℓ it is sufficient to determine any vector perpendicular to $\mathbf{P}_1$ and $\mathbf{P}_2$. In particular, the cross product gives a vector perpendicular to two given vectors, thus $\ell = \mathbf{P}_1 \times \mathbf{P}_2$ (or any multiple of $\mathbf{P}_1 \times \mathbf{P}_2$). Hence, the equation of the line through two points can be determined by taking the "cross product" of the homogeneous coordinates of the points.

Example 2.20

The line ℓ passing through $(0, 5)$ and $(6, -7)$ satisfies

$$\ell \cdot (0, 5, 1) = 0 \quad \text{and} \quad \ell \cdot (6, -7, 1) = 0\ .$$

Hence

$$\ell = (0, 5, 1) \times (6, -7, 1) = (12, 6, -30)$$

giving the line $12x + 6y - 30 = 0$.

To Determine the Point of Intersection of Two Lines

Suppose $\mathbf{P}$ is the point of intersection of two lines ℓ_1 and ℓ_2. Then $\mathbf{P}$ is a point on both lines and (2.13) yields

$$\ell_1 \cdot \mathbf{P} = 0 \quad \text{and} \quad \ell_2 \cdot \mathbf{P} = 0\ .$$

Hence $\mathbf{P}$ is a vector perpendicular to both ℓ_1 and ℓ_2, and hence it is sufficient to take $\mathbf{P} = \ell_1 \times \ell_2$ (or any multiple of it). The cross product yields the homogeneous coordinates of the point of intersection.

Example 2.21

The point $\mathbf{P}$ of intersection of the lines $x - 7y + 8 = 0$ and $3x - 4y + 1 = 0$ satisfies

$$(1, -7, 8) \cdot \mathbf{P} = 0 \quad \text{and} \quad (3, -4, 1) \cdot \mathbf{P} = 0 \ .$$

Hence

$$\mathbf{P} = (1, -7, 8) \times (3, -4, 1) = (25, 23, 17) \ .$$

The Cartesian coordinates of the intersection point are $(25/17, 23/17)$.

Example 2.22

The point $\mathbf{P}$ of intersection of the lines $2x - 5y = 0$ and $2x - 5y + 3 = 0$ has homogeneous coordinates

$$\mathbf{P} = (2, -5, 0) \times (2, -5, 3) = (-15, -6, 0) \ .$$

The point of intersection $(-15, -6, 0)$ is a point at infinity since the lines are parallel.

EXERCISES

2.28. Determine the line passing through $(1, 3)$ and $(4, -2)$.

2.29. Determine the point of intersection of the lines $x - 3y + 7 = 0$ and $4x + 3y - 5 = 0$.

2.30. The methods used to determine the line through two distinct points and the point of intersection of two lines both involve the cross product. This is due to the duality between points and lines in the plane which relates results about points and lines to a dual result about lines and points. For example, the property "points r_1, r_2, and r_3 are *collinear* if and only if $r_1 \cdot (r_2 \times r_3) = 0$" has the *dual* property "lines ℓ_1, ℓ_2, and ℓ_3 are *concurrent* if and only if $\ell_1 \cdot (\ell_2 \times \ell_3) = 0$". Investigate further the property of duality [24, pp78–80].

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